# Genus 3 covers of elliptic curves

## Davide Lombardo, Elisa Lorenzo-García, Jeroen Sijsling

28 March 2017

Genus 3 covers

28 March 2017 1 / 9

$$C \xrightarrow{d-to-1} E$$

with C of genus 3 and E an elliptic curve



Image: Image:

$$C \xrightarrow{d-to-1} E$$

with C of genus 3 and E an elliptic curve (everything defined over  $\overline{\mathbb{Q}}$ )



$$C \xrightarrow{d-to-1} E$$

with C of genus 3 and E an elliptic curve (everything defined over  $\overline{\mathbb{Q}}$ ) Up to isogeny, we have either

• Jac(
$$C$$
) ~  $E \times E_2 \times E_3$ , or

$$C \xrightarrow{d-to-1} E$$

with C of genus 3 and E an elliptic curve (everything defined over  $\overline{\mathbb{Q}}$ ) Up to isogeny, we have either

• Jac(
$$C$$
) ~  $E \times E_2 \times E_3$ , or

**2**  $\operatorname{Jac}(C) \sim E \times \operatorname{Jac}(X)$  with X of genus 2

• Decide in which case we are



• • • • • • • •

æ

- Decide in which case we are
- Find  $E_2, E_3$  or X.

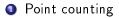


- I A P

- Decide in which case we are
- Find  $E_2, E_3$  or X.

### Remark

Finding  $E_2, E_3$  is as hard as finding E, and we know how to do that.





- Oint counting
- 2 Endomorphism verification



- Oint counting
- 2 Endomorphism verification
- O Prym-like varieties



#### Suppose

$$C: y^4 - h(x,z)y^2 + f(x,z)g(x,z) = 0$$



#### Suppose

$$C: y^4 - h(x,z)y^2 + f(x,z)g(x,z) = 0$$

and

$$E: y^2 - h(x,z)y + f(x,z)g(x,z) = 0.$$

#### Suppose

$$C: y^4 - h(x,z)y^2 + f(x,z)g(x,z) = 0$$

and

$$E: y^2 - h(x, z)y + f(x, z)g(x, z) = 0.$$

If  $\operatorname{Aut}(C) = \mathbb{Z}/2\mathbb{Z}$  one can explicitly write down a genus 2 curve X such that  $\operatorname{Jac}(C) \sim E \times \operatorname{Jac}(X)$ .

#### Suppose

$$C: y^4 - h(x,z)y^2 + f(x,z)g(x,z) = 0$$

and

$$E: y^2 - h(x, z)y + f(x, z)g(x, z) = 0.$$

If  $\operatorname{Aut}(C) = \mathbb{Z}/2\mathbb{Z}$  one can explicitly write down a genus 2 curve X such that  $\operatorname{Jac}(C) \sim E \times \operatorname{Jac}(X)$ . X is defined over the same field as C.

• Suppose  $C \to E$  is Galois with "large" automorphism group – i.e.  $D_4, Q_8, S_3$ . Then Jac(C) is the product of three elliptic curves.

- Suppose  $C \to E$  is Galois with "large" automorphism group i.e.  $D_4, Q_8, S_3$ . Then Jac(C) is the product of three elliptic curves.
- $\bullet\,$  This is not necessarily the case if the automorphism group of the covering is  $\mathbb{Z}/2\mathbb{Z}$

- Suppose  $C \to E$  is Galois with "large" automorphism group i.e.  $D_4, Q_8, S_3$ . Then Jac(C) is the product of three elliptic curves.
- $\bullet\,$  This is not necessarily the case if the automorphism group of the covering is  $\mathbb{Z}/2\mathbb{Z}$
- $\bullet$  When the group is  $\mathbb{Z}/3\mathbb{Z},$  the abelian surface has QM

## The elliptic curve E is canonically an abelian subvariety of Jac(C).



The elliptic curve E is canonically an abelian subvariety of Jac(C). There is a canonical abelian surface  $\iota_A : A \hookrightarrow Jac(C)$  such that  $A \times E \to Jac(C)$  is an isogeny.

The elliptic curve E is canonically an abelian subvariety of Jac(C). There is a canonical abelian surface  $\iota_A : A \hookrightarrow Jac(C)$  such that  $A \times E \to Jac(C)$  is an isogeny.

### Question

Let  $\Theta$  be the theta divisor of Jac(C). What is the degree of the polarization  $\iota_A^* \Theta$ ?

Partial answer	
There is a	d-isogeny $A  ightarrow A'$ with $A'$ principally polarized

• • • • • • • •

### Partial answer

There is a d-isogeny  $A \rightarrow A'$  with A' principally polarized (hence a Jacobian or a product of two elliptic curves).



#### Partial answer

There is a (non-canonical!) *d*-isogeny  $A \rightarrow A'$  with A' principally polarized (hence a Jacobian or a product of two elliptic curves).



#### Partial answer

There is a (non-canonical!) *d*-isogeny  $A \rightarrow A'$  with A' principally polarized (hence a Jacobian or a product of two elliptic curves).

### Question

Is the isogeny defined over the same field as  $C \rightarrow E$ ?

### • From $C \rightarrow E$ determine a period matrix of C, hence of A



- From  $C \rightarrow E$  determine a period matrix of C, hence of A
- Determine an isogeny  $A \rightarrow A'$  with A' principally polarized



- From  $C \rightarrow E$  determine a period matrix of C, hence of A
- Determine an isogeny  $A \rightarrow A'$  with A' principally polarized
- Write down the period matrix of A' = Jac(X)

- From  $C \rightarrow E$  determine a period matrix of C, hence of A
- Determine an isogeny  $A \rightarrow A'$  with A' principally polarized
- Write down the period matrix of A' = Jac(X)
- Reconstruct X from A' (Guàrdia)